

Signed Posets*

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We define a new object, called a *signed poset*, that bears the same relation to the *hyperoctahedral group* B_n (i.e., signed permutations on n letters), as do posets to the *symmetric group* S_n . We then prove hyperoctahedral analogues of the following results: (1) the generating function results from the theory of P -partitions; (2) the fundamental theorem of finite distributive lattices (or Birkhoff's theorem) relating a poset to its distributive lattice of order ideals; (3) the edgewise-lexicographic shelling of upper-semimodular lattices; (4) MacMahon's calculation of the distribution of the major index for permutations. © 1993 Academic Press, Inc.

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1. INTRODUCTION: POSETS VIEWED IN TERMS OF A ROOT SYSTEM

In this section, we review some basic notions about posets and rephrase them in terms of the *root system* associated to the symmetric group S_n . Although this rephrasing may seem unnecessary, it will motivate our subsequent definitions of hyperoctahedral analogues.

DEFINITION. Let e_i denote the i th standard basis vector in \mathbf{R}^n . The *root system* for S_n is the set of vectors

$$\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n\} \subseteq \mathbf{R}^n,$$

* This material appeared as part of the author's doctoral thesis at MIT under the supervision of R. Stanley.

with *positive roots*

$$\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\},$$

and *simple roots*

$$\Pi = \{e_i - e_{i+1} : 1 \leq i < n\}.$$

We will have need of only a few of the properties of root systems, but the interested reader may find out more about them in [Bo]. The few properties of Φ , Φ^+ , Π that we need (and that are easy to check) are:

1. $\Phi = \Phi^+ \amalg -\Phi^+$, where $-\Phi^+ = \{-\alpha : \alpha \in \Phi^+\}$ and “ \amalg ” denotes disjoint union.

2. If for $S \subseteq \Phi$ we denote by

$$\bar{S}^{\text{PLC}} = \left\{ \alpha \in \Phi : \alpha = \sum_{\beta \in S} c_\beta \beta \text{ for some } c_\beta \geq 0 \right\}$$

the set of positive linear combination of elements of S that lie in Φ , then we have that $\Phi^+ = \bar{\Pi}^{\text{PLC}}$.

3. If we consider S_n as acting on \mathbf{R}^n by permuting coordinates, this action preserves Φ .

Given a poset P with n elements, without loss of generality we may assume P is a partial order $<_P$ on the set $\{1, 2, \dots, n\}$ (this is sometimes called a *labelled poset*). We will identify such a partial order with a subset $P \subseteq \Phi$ in the following way: $e_i - e_j \in P$ if and only if $i <_P j$, i.e.,

$$P = \{e_i - e_j : 1 \leq i \neq j \leq n, i <_P j\}.$$

We then have the following

PROPOSITION 1.1. *The above identification gives a one-to-one correspondence between partial orders $<_P$ on $\{1, 2, \dots, n\}$ and subsets $P \subseteq \Phi$ satisfying*

1. $\alpha \in P \Rightarrow -\alpha \notin P$,
2. $\bar{P}^{\text{PLC}} = P$.

Proof. Property 1 corresponds to *antisymmetry* of $<_P$:

$$\begin{aligned} i <_P j &\Rightarrow j \not<_P i \\ e_i - e_j \in P &\Rightarrow e_j - e_i \notin P. \end{aligned}$$

Property 2 corresponds to transitivity of $<_P$:

$$i <_P j, j <_P k \Rightarrow i <_P k$$

$$e_i - e_j, e_j - e_k \in P \Rightarrow (e_i - e_j) + (e_j - e_k) = e_i - e_k \in P.$$

Irreflexivity of $<_P$ corresponds to the fact that $0 \notin P$. ■

The notion of an order ideal of a poset fits nicely with the above correspondence.

DEFINITION. Let P be a partial order on $\{1, 2, \dots, n\}$. Then $I \subseteq \{1, 2, \dots, n\}$ is an *ideal* of P if $j \in I$ and $i <_P j$ implies $i \in I$. We can rephrase this in terms of the above correspondence if we identify subsets $I \subseteq \{1, 2, \dots, n\}$ with their *characteristic vector* (i.e., the vector $(\varepsilon_1, \dots, \varepsilon_n) \in \{1, 0\}^n$ with $\varepsilon_i = 1$ if $i \in I$ and 0 else). We have that I is an ideal of P exactly when $\langle \alpha, I \rangle \geq 0$ for all $\alpha \in P$ (where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbf{R}^n).

One can also define the poset $J(P)$ of ideals of P ordered under inclusion. This ordering corresponds to the componentwise partial order on the characteristic vectors, with $1 > 0$ in each component.

The basic result about $J(P)$ is known as the Fundamental Theorem Of Finite Distributive Lattices, or Birkhoff's theorem (see [St1, Theorem 3.4.1] for a proof):

THEOREM 1.2. $J(P)$ is a finite distributive lattice, i.e., it satisfies the distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in J(P).$$

Furthermore, every finite distributive lattice L is isomorphic to $J(P)$ for some poset P , which is uniquely defined up to poset isomorphism.

Note that the notion of poset isomorphism is quite natural to define under our correspondence: two posets P_1, P_2 are isomorphic if there exists some permutation $\pi \in S_n$ such that $\pi P_1 = P_2$.

Another notion which can be rephrased quite naturally is that of a P -partition, a common generalization of partitions, partitions into distinct parts, compositions, plane partitions, and column-strict tableaux (for more about P -partitions, see [St2]).

DEFINITION. Given a partial order P on $\{1, 2, \dots, n\}$, a P -partition is a function $f: \{1, 2, \dots, n\} \rightarrow \mathbf{N}$ satisfying

$$\begin{aligned} f(i) &\geq f(j) && \text{if } i <_P j \\ f(i) &> f(j) && \text{if } i <_P j \text{ and } i > j. \end{aligned}$$

Rephrasing this in terms of our correspondence, a P -partition is a vector $f \in \mathbb{N}^n$ satisfying

$$\begin{aligned} \langle \alpha, f \rangle &\geq 0 && \text{for all } \alpha \in P \\ \langle \alpha, f \rangle &> 0 && \text{for all } \alpha \in P \cap -\Phi^+. \end{aligned}$$

In subsequent sections, we define hyperoctahedral analogues of all of these concepts and prove analogues of some of the basic results known about them.

2. SIGNED POSETS

In this section we introduce and give examples of signed posets, our hyperoctahedral analogues of posets.

DEFINITION. Let B_n denote the *hyperoctahedral group* or the group of *signed permutations*, i.e., all permutations and sign changes of the coordinates in \mathbb{R}^n . The *root system* for B_n is the set of vectors

$$\Phi = \{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\},$$

with *positive roots*

$$\Phi^+ = \{+e_i : 1 \leq i \leq n\} \cup \{+e_i + e_j, +e_i - e_j : 1 \leq i < j \leq n\},$$

and *simple roots*

$$\Pi = \{e_i - e_{i+1} : 1 \leq i < n\} \cup \{+e_n\}.$$

One can easily check that Properties 1–3 that held for Φ and S_n in Section 1 also hold here for Φ and B_n .

Motivated by Proposition 1.1, we define a *signed poset* P (on n elements) to be a subset $P \subseteq \Phi$ satisfying

1. $\alpha \in P \Rightarrow -\alpha \notin P$
2. $\bar{P}^{\text{PLC}} = P$.

We will say that two signed posets P_1, P_2 are *isomorphic* (written $P_1 \cong P_2$) if there exists a signed permutation $w \in B_n$ such that $wP_1 = P_2$.

It is convenient to have a pictorial representation of a signed poset P , so we define its *signed digraph* $D(P)$ as follows. $D(P)$ is a graph on vertex set $\{1, 2, \dots, n\}$ with certain labelled edges and labelled loops attached to the vertices:

If $+e_i \in P$, attach a loop like this: $(i) \oplus$

If $-e_i \in P$, attach a loop like this: $(i) \ominus$

If $+e_i + e_j \in P$, attach an edge like this: $(i) \oplus \text{---} \oplus (j)$

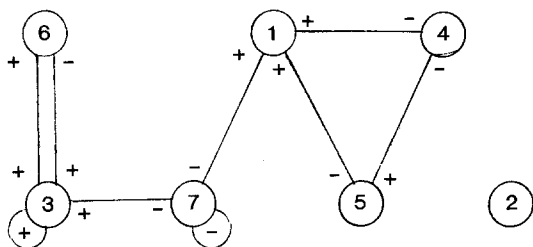
If $-e_i - e_j \in P$, attach an edge like this: $(i) \ominus \text{---} \ominus (j)$

If $+e_i - e_j \in P$, attach an edge like this: $(i) \oplus \text{---} \ominus (j)$

EXAMPLE. The set $P = \{+e_1 - e_2, +e_1 + e_2\}$ is *not* a signed poset, since

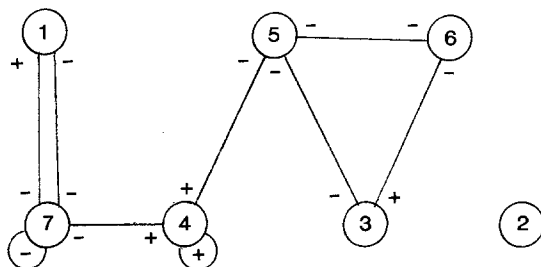
$$+e_1 = \frac{1}{2}(+e_1 - e_2) + \frac{1}{2}(+e_1 + e_2) \in \bar{P}^{\text{PLC}}$$

but $+e_1 \notin P$, violating axiom 2 (the “transitivity” axiom). Neither is $P = \{+e_1 + e_2, -e_1 - e_2\}$, since it violates axiom 1 (the “antisymmetry” axiom). Figure 1 shows two examples of signed posets P_1 , P_2 and their signed digraphs, along with the isomorphism $P_1 \cong P_2$.



$$P_1 = \{+e_3, -e_7, +e_1 - e_4, +e_1 - e_5, +e_1 - e_7$$

$$+e_3 + e_6, +e_3 - e_6, +e_3 - e_7, -e_4 + e_5\}$$



$$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -5 & +2 & -7 & +6 & +3 & +1 & -4 \end{pmatrix} P_1$$

FIG. 1. An example of two isomorphic signed posets.

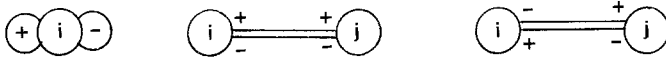


FIG. 2. Configurations not allowed.

Remark. In [Re], signed posets are called B_n -parsets because they are a special case of the notion of a *parset* defined for arbitrary root systems. The terminology “signed poset” seems more in line with other hyperoctahedral terminology such as “signed permutation” and “signed sets.” We also hope that the terminology “signed digraph” is consistent with the theory of *signed graphs* developed by Zaslavsky [Za].

The axioms of signed posets dictate that certain configurations of edges and loops in $D(P)$ cannot occur, and certain configurations imply the existence of more edges and loops. These rules are summarized in Figs. 2 and 3.

Remark. Since $\bar{P}^{\text{PLC}} = P$, one might try to simplify $D(P)$ by eliminating those edges and loops which are implied by others. It is in fact true that for any signed poset P , there is a unique minimal subset $H \subseteq P$ such that $\bar{H}^{\text{PLC}} = P$ (this is a reflection of the easy-to-check fact that \cdot^{PLC} is a *convex closure*; see [EJ, Theorem 2.1]). Thus if we were to include only the loops and edges that correspond to elements of H , we would obtain the analogue

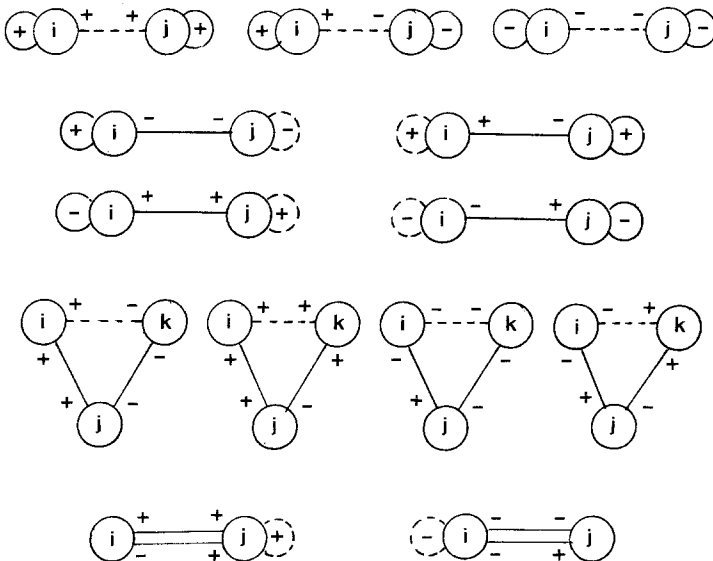


FIG. 3. Configurations implying more edges (implied edges shown dotted).

of the *Hasse diagram* of P . However, we feel that these Hasse diagrams are harder to understand than $D(P)$ (because “transitivity” is more complicated for signed posets than it was for posets).

One further notion for which we need an analogue is that of a subposet.

DEFINITION. Let P be a signed poset (on n elements) and $T \subseteq \{1, 2, \dots, n\}$. The *induced signed subposet* of P on T is the signed poset P_T on $\#T$ elements consisting of only those roots in P whose nonzero coordinates lie in T (strictly speaking, we should also re-index the coordinates in T to be $\{1, 2, \dots, \#T\}$).

EXAMPLE. For the signed poset

$$P_1 = \{+e_3, -e_7, +e_1 - e_7, +e_1 - e_4, +e_1 - e_5, \\ +e_3 + e_6, +e_3 - e_6, +e_3 - e_7, -e_4 + e_5\},$$

from Fig. 1, and $T = \{2, 3, 4, 6\}$, we have that (without re-indexing)

$$P_{1T} = \{+e_3, +e_3 + e_6, +e_3 - e_6\}.$$

3. P -PARTITIONS, GENERATING FUNCTIONS, AND $J(P)$

In this section, we define P -partitions for signed posets P and investigate some of their various generating functions and counting functions. We also introduce the poset $J(P)$ of ideals of P . Our notation and exposition are intended to parallel those of [St2]. In fact, by embedding posets as signed posets in a certain fashion, we can arrange that all of our definitions and results when particularized to these embedded posets yield a result analogous to that of [St2]. This embedding is done as follows:

DEFINITION. Given a poset P on $\{1, 2, \dots, n\}$, we define its *positive embedding* P^+ by

$$P^+ = \{+e_i - e_j : 1 \leq i \neq j \leq n, i <_P j\} \\ \cup \{+e_i + e_j : 1 \leq i < j \leq n\} \cup \{+e_i : 1 \leq i \leq n\}.$$

One can see readily that P^+ is a signed poset.

Motivated by our rephrasing of the definition of P -partitions, ideals, and $J(P)$ given in Section 1, we make the following definitions.

DEFINITION. For P a signed poset (on n elements), a P -partition is a vector $f \in \mathbb{Z}^n$ satisfying

$$\begin{aligned}\langle \alpha, f \rangle &\geq 0 & \text{for all } \alpha \in P \\ \langle \alpha, f \rangle &> 0 & \text{for all } \alpha \in P \cap -\Phi^+.\end{aligned}$$

We denote by $\mathcal{A}(P)$ the set of all P -partitions. For $m \in \mathbb{N}$, define

$$\mathcal{A}(P; m) = \{(f_1, \dots, f_n) \in \mathcal{A}(P) : |f_i| \leq m \forall i\}.$$

Define $F(P, x)$ to be the formal power series in variables $x_1, x_{-1}, \dots, x_n, x_{-n}$ given by

$$F(P, x) = \sum_{f \in \mathcal{A}(P)} x^f,$$

where $x^f = \prod_{i=1}^n x_{\text{sgn}(f_i) \cdot i}^{|f_i|}$ (e.g., $x^{(-5, 1, 3, -2)} = x_{-1}^5 x_2^1 x_3^3 x_{-4}^2$). Let $U_m(P, x)$ be the formal power series in one variable x defined by

$$U_m(P, x) = \sum_{f \in \mathcal{A}(P; m)} x^{|f_1| + \dots + |f_n|}$$

and

$$U(P, x) = \lim_{m \rightarrow \infty} U_m(P, x) = \sum_{f \in \mathcal{A}(P)} x^{|f_1| + \dots + |f_n|} = F(P, x) \big|_{x_{\pm i} \rightarrow x}.$$

Define a poset

$$J(P) = \{f \in \{+1, -1, 0\}^n : \langle \alpha, f \rangle \geq 0 \forall \alpha \in P\}$$

with partial order inherited from $\{+1, -1, 0\}^n$ by setting $+1 > 0$, $-1 > 0$ and extending componentwise. We will call an element $I \in J(P)$ an *ideal* of P .

EXAMPLE. Let $P = \{+e_2 - e_1, +e_2\}$. Then

$$\mathcal{A}(P) = \{(f_1, f_2) \in \mathbb{Z}^2 : f_2 > f_1, f_2 \geq 0\}$$

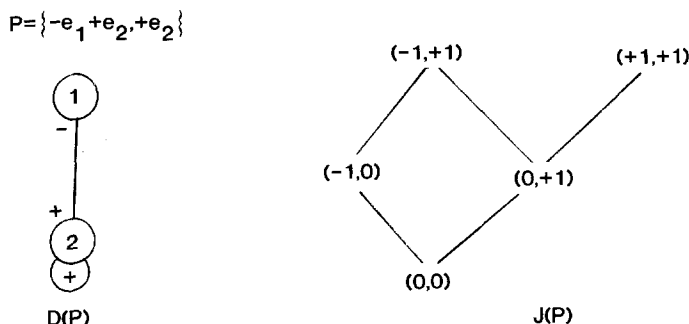
and

$$\begin{aligned}F(P, x) &= \sum_{f_2 > f_1, f_2 \geq 0} x^f = \sum_{f_2 > f_1 \geq 0} x^f + \sum_{f_2 \geq 0 > f_1} x^f \\ &= \frac{x_2}{(1-x_2)(1-x_1x_2)} + \frac{x_{-1}}{(1-x_{-1})(1-x_2)}\end{aligned}$$

and thus

$$U(P, x) = \frac{x}{(1-x)^2} + \frac{x}{(1-x)(1-x^2)} = \frac{x^2 + 2x}{(1-x)(1-x^2)}.$$

$D(P)$ and $J(P)$ are shown in Fig. 4.

FIG. 4. An example of $D(P)$ and $J(P)$.

Our first step toward nice expressions for our generating functions is to relate P -partitions to chains in $J(P)$. The next definition gives us a way of relating a vector in \mathbb{Z}^n to a chain in $\{+1, -1, 0\}^n$.

DEFINITION. Given $f \in \mathbb{Z}^n$, let $\{|f_i|\}_{i=1, \dots, n} = \{n_1, \dots, n_k\}$ with $n_1 > \dots > n_k$. Define a chain $c(f)$ of vectors $I_1 < \dots < I_k$ in $\{+1, -1, 0\}^n$ by

$$I_i(j) = \begin{cases} \text{sgn}(f_j) & \text{if } |f_j| \geq n_i \\ 0 & \text{else} \end{cases}$$

for $j = 1, \dots, n$. For example, if $f = (-5, +1, +3, -2, -3)$ then $\{|f_i|\}_{i=1,2,3,4,5} = \{5, 3, 2, 1\}$ and $c(f)$ is

$$\begin{aligned} (-1, 0, 0, 0, 0) &\leq (-1, 0, +1, 0, -1) \leq (-1, 0, +1, -1, -1) \\ &\leq (-1, +1, +1, -1, -1). \end{aligned}$$

One way to visualize this is as follows. Given f , make a histogram that has the coordinates f_i along the bottom, and a column of ± 1 's (depending on $\text{sgn}(f_i)$) of height $|f_i|$ above each f_i , filling in zeroes elsewhere. Then I_1, \dots, I_k are the set of (distinct) rows read from top to bottom. For instance, in the example above, we have

-1	0	0	0	0	← I_1
-1	0	0	0	0	
-1	0	+1	0	-1	← I_2
-1	0	+1	-1	-1	← I_3
-1	+1	+1	-1	-1	← I_4
<hr/>					
-5	+1	+3	-2	-3	← f

Given c a chain $I_1 \leq \dots \leq I_k$ in $J(P)$, we will say c is P -compatible if it satisfies the following condition: for $0 \leq i < k$, when we restrict I_{i+1} to the set S_i of coordinates where it differs from I_i (setting $I_0 = \emptyset$), we get a vector in $\mathcal{A}(P_{S_i}; 1)$, where P_{S_i} is the induced signed subposet of P on S_i .

The next proposition gives the basic relation between $\mathcal{A}(P)$ and $J(P)$.

PROPOSITION 3.1. *A vector f is in $\mathcal{A}(P)$ if and only if $c(f)$ is a P -compatible chain in $J(P) - \hat{0}$.*

Proof. A look at the histogram picture should convince one that $\langle \alpha, f \rangle \geq 0 \forall \alpha \in P$ if and only if $\langle \alpha, I_i \rangle \geq 0 \forall \alpha \in P, \forall i$, i.e., if and only if $c(f)$ is a chain in $J(P)$. It then remains to show that $\langle \alpha, f \rangle > 0 \forall \alpha \in P \cap -\Phi^+$ if and only if $c(f)$ is P -compatible. One can check this for the various cases of $\alpha \in P \cap -\Phi^+$, i.e., α of the form $-e_j$, $-e_j - e_k$, or $+e_j - e_k$ for $j > k$. We illustrate this for the second case; the others are similar. If $\alpha = -e_j - e_k$ then $\langle \alpha, f \rangle > 0$ if and only if $f_j + f_k < 0$. This is equivalent to the condition that whenever I_{i+1} and I_i differ in coordinates j and k we have $I_{i+1}(j) = I_{i+1}(k) = -1$, which is one of the conditions for $c(f)$ to be P -compatible. ■

COROLLARY 3.2.

$$F(P, x) = \sum \frac{x^{I_1} \dots x^{I_k}}{(1 - x^{I_1}) \dots (1 - x^{I_k})},$$

where the above sum ranges over all P -compatible chains $I_1 < \dots < I_k$ in $J(P) - \hat{0}$.

Proof.

$$\begin{aligned} F(P, x) &= \sum_{f \in \mathcal{A}(P)} x^f \\ &= \sum_{\substack{P\text{-compatible} \\ I_1 < \dots < I_k \in J(P)}} \sum_{\substack{f \in \mathbb{Z}^n \\ c(f) = I_1 < \dots < I_k}} x^f \\ &= \sum_{\substack{P\text{-compatible} \\ I_1 < \dots < I_k \in J(P)}} \frac{x^{I_1} \dots x^{I_k}}{(1 - x^{I_1}) \dots (1 - x^{I_k})}, \end{aligned}$$

where the second equality comes from Proposition 3.1. ■

EXAMPLE. For $P = \{+e_2 - e_1, +e_2\}$ as before, the P -compatible chains in $J(P)$ are

$$\begin{aligned} &(-1, 0), \quad (0, +1), \quad (-1, +1) \\ &(-1, 0) < (-1, +1), \quad (0, +1) < (-1, +1), \quad (0, +1) < (+1, +1) \end{aligned}$$

and thus according to the previous proposition,

$$\begin{aligned} F(P, x) = & \frac{x_{-1}}{(1-x_{-1})} + \frac{x_2}{(1-x_2)} + \frac{x_{-1}x_2}{(1-x_{-1}x_2)} \\ & + \frac{x_{-1}-x_2}{(1-x_{-1})(1-x_{-1}x_2)} + \frac{x_2 \cdot x_{-1}x_2}{(1-x_2)(1-x_{-1}x_2)} \\ & + \frac{x_2 \cdot x_1x_2}{(1-x_2)(1-x_1x_2)}. \end{aligned}$$

With a great deal of manipulation, one can check that this agrees with our earlier calculation of $F(P, x)$.

Most of our expressions for generating functions will follow from our next result, which is the analogue of the central result on P -partitions [St2, Theorem 6.2]. Although a more general statement was proven in [Re], we include the proof in this case for the sake of completeness.

DEFINITION. Given P a signed poset (on n elements), let its *Jordan-Hölder set* $\mathcal{L}(P)$ be the set $\{w \in B_n : P \subseteq w\Phi^+\}$.

THEOREM 3.3.

$$\mathcal{A}(P) = \coprod_{w \in \mathcal{L}(P)} \mathcal{A}(w\Phi^+).$$

Proof. (cf. [Ge, Theorem 1]). We proceed by induction on

$$r = \#\{\alpha \in \Phi^+ : \alpha \notin P, -\alpha \notin P\}.$$

$r=0$: In this case, it suffices to show that $P = w\Phi^+$ for some $w \in B_n$ (since then one easily sees that $\mathcal{L}(P) = \{w\}$). Since $r=0$, for each i , we have either $+e_i \in P$ or $-e_i \in P$. Thus we can find an element $w' \in B_n$ such that $+e_i \in P' = w'P$ for all i . Since $\overline{P'}^{\text{PLC}} = P'$, this implies that $+e_i + e_j \in P'$ for all $i < j$, and hence $P' = Q^+$ for some poset Q on $\{1, 2, \dots, n\}$. Since either $+e_i - e_j \in P'$ or $+e_j - e_i \in P'$ for all $i < j$, Q must be a total order on $\{1, 2, \dots, n\}$. Thus we can find some permutation $w'' \in B_n$ such that $P'' = w''P'$ has $+e_i - e_j \in P''$ if and only if $i < j$. Then $P'' = \Phi^+$ and hence $P = (w''w')^{-1}\Phi^+$, as we wanted.

$r>0$: Assume $\alpha, -\alpha \notin P$, and let $P_\alpha = \overline{P \cup \{\alpha\}}^{\text{PLC}}$. We claim P_α is a

signed poset, i.e., it also satisfies the "antisymmetry" axiom. To see this, suppose not, i.e., let $\beta, -\beta \in P_\alpha$. Then we must have

$$\begin{aligned}\beta &= a\alpha + \sum a_i \alpha_i \\ -\beta &= b\alpha + \sum b_i \alpha_i\end{aligned}$$

for some $a_i, b_i \geq 0$, $a, b > 0$, and $\alpha_i \in P$. Adding these equations, and dividing by $a + b$, yields

$$-\alpha = \sum \frac{1}{a+b} (a_i + b_i) \alpha_i$$

and hence $-\alpha \in P$, a contradiction. Similarly we can form the signed poset $P_{-\alpha}$. We then have

$$\begin{aligned}\mathcal{A}(P) &= \mathcal{A}(P_\alpha) \amalg \mathcal{A}(P_{-\alpha}) \\ \mathcal{L}(P) &= \mathcal{L}(P_\alpha) \amalg \mathcal{L}(P_{-\alpha}).\end{aligned}$$

The first equality holds because any $f \in \mathcal{A}(P)$ satisfies either $\langle \alpha, f \rangle \geq 0$ or $\langle -\alpha, f \rangle > 0$. The second equality holds because any $w \in \mathcal{L}(P)$ satisfies either $\alpha \in w\Phi^+$ or $-\alpha \in w\Phi^+$. Thus by induction on r , we are done. ■

In order to make use of this theorem, we need only understand the sets $\mathcal{A}(w\Phi^+)$ for $w \in B_n$ more fully. For this, we require the notion of a descent of w .

DEFINITION. Given

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & & w_n \end{pmatrix} \in B_n,$$

its (right) descent set $D(w)$ is defined as follows: numbering the elements of Π as

$$\pi_1 = +e_1 - e_2, \pi_2 = +e_2 - e_3, \dots, \pi_{n-1} = +e_{n-1} - e_n, \pi_n = +e_n,$$

we define $D(w) = \{i : w(\pi_i) \in -\Phi^+\}$.

For $1 \leq i \leq n$, let $\delta_i(w)$ be the characteristic vector of the signed subset $\{w_1, w_2, \dots, w_i\}$ (i.e., $\delta_i(w)$ is the vector $(\varepsilon_1, \dots, \varepsilon_n) \in \{+1, -1, 0\}^n$ with $\varepsilon_i = +1, -1$, or 0 depending on whether i appears in $\{w_1, w_2, \dots, w_i\}$ with a $+$, with a $-$, or not at all). We then define two chains $c_R(w), c_E(w)$ in $\{+1, -1, 0\}^n$ associated to w as

$$\begin{aligned}c_R(w) &= \delta_{i_1} < \cdots < \delta_{i_k}, \quad \text{where } D(w) = \{i_1 < \cdots < i_k\} \\ c_E(w) &= \delta_1 < \cdots < \delta_n.\end{aligned}$$

EXAMPLE. Let $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & +1 & -4 & +5 & -2 \end{pmatrix}$. Then $D(w) = \{1, 3, 5\}$ and

$$c_R(w) = (0, 0, -1, 0, 0) < (+1, 0, -1, -1, 0) < (+1, -1, -1, -1, +1)$$

$$c_E(w) = (0, 0, -1, 0, 0) < (+1, 0, -1, 0, 0) < (+1, 0, -1, -1, 0)$$

$$< (+1, 0, -1, -1, +1) < (+1, -1, -1, -1, +1).$$

The following proposition tells us what we need to know about $\mathcal{A}(w\Phi^+)$. Its proof is a straightforward unravelling of the various definitions, which is left to the reader.

PROPOSITION 3.4. *Let*

$$w = \begin{pmatrix} 1 & \cdots & b \\ w_1 & \cdots & w_n \end{pmatrix} \in B_n$$

and $f \in \mathbb{Z}^n$. Then the following are equivalent:

1. $f \in \mathcal{A}(w\Phi^+)$.
2. $\text{sgn}(w_i) = \text{sgn}(f_{|w_i|})$ and

$$|f_{|w_1|}| \sim_1 \cdots \sim_{n-1} |f_{|w_n|}| \sim_n 0,$$

where " \sim_i " = " \geq " if $s_i \in D(w)$ and " \sim_i " = " $>$ " else.

3. $c_R(w) \subseteq c(f) \subseteq c_E(w)$.

EXAMPLE. If $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & +1 & -4 & +5 & -2 \end{pmatrix}$, then $D(w) = \{(12), (34), (\begin{smallmatrix} 5 \\ -5 \end{smallmatrix})\}$, and $f \in \mathcal{A}(w\Phi^+)$ if and only if $f_2, f_4 \geq 0, f_1, f_3, f_5 \leq 0$ and

$$|f_3| > |f_1| \geq |f_4| > |f_5| \geq |f_2| > 0.$$

PROPOSITION 3.5.

$$F(P, x) = \sum_{w \in \mathcal{L}(P)} \frac{\prod_{s_i \in D(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{i=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_i})}$$

and hence

$$U(P, x) = \sum_{w \in \mathcal{L}(P)} \frac{x^{\text{maj}(w)}}{(1-x)(1-x^2) \cdots (1-x^n)},$$

where $\text{maj}(w) = \sum_{s_i \in D(w)} i$.

Proof. We have

$$F(P, x) = \sum_{f \in AP} x^f = \sum_{w \in \mathcal{L}(P)} \sum_{f \in \mathcal{A}(w\Phi^+)} x^f$$

and from the previous proposition, one can see that

$$\sum_{f \in \mathcal{A}(w\Phi^+)} x^f = \frac{\prod_{s_i \in D(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{i=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_i})}. \quad \blacksquare$$

To write down expressions for the other generating functions, we need a little terminology.

DEFINITION. Let

$$W(P, x) = \sum_{w \in \mathcal{L}(P)} x^{\text{maj}(w)}$$

be the numerator in the above expression for $U(P, x)$, and for $0 \leq s \leq n$ let

$$W_s(P, x) = \sum_{\substack{w \in \mathcal{L}(P) \\ \#D(w) = s}} x^{\text{maj}(w)}$$

so that $W(P, x) = \sum_{s=0}^n W_s(P, x)$. The Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_x$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{(1-x^n)(1-x^{n-1}) \cdots (1-x^{n-k+1})}{(1-x^k)(1-x^{k-1}) \cdots (1-x)}.$$

PROPOSITION 3.6.

1. $U_m(P, x) = \sum_{s=0}^n \begin{bmatrix} n+s \\ n \end{bmatrix}_x W_{m-s}(P, x),$
2. $\sum_{m \geq 0} U_m(P, x) q^m = \frac{\sum_{s=0}^n q^s W_s(P, x)}{(1-q)(1-qx) \cdots (1-qx^n)}.$

Proof.

1. We have

$$U_m(P, x) = \sum_{f \in \mathcal{A}(P; m)} x^{|f_1| + \cdots + |f_n|} = \sum_{w \in \mathcal{L}(P)} \sum_{f \in \mathcal{A}(w\Phi^+; m)} x^{|f_1| + \cdots + |f_n|}.$$

By Proposition 3.4, $f \in \mathcal{A}(w\Phi^+; m)$ if and only if $\text{sgn}(w_i) = \text{sgn}(f_{|w_i|})$ and

$$m \geq |f_{|w_1|}| \geq \cdots \geq |f_{|w_n|}| \geq 0$$

with strict inequalities at the descents of w . If we let

$$\lambda_i = |f_{|w_i|}| - \#(D(w) \cap \{s_i, s_{i+1}, \dots, s_n\})$$

then we have

$$|f_1| + \cdots + |f_n| = \text{maj}(w) + \lambda_1 + \cdots + \lambda_n$$

and

$$m - \#D(w) \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

Thus

$$\begin{aligned} U_m(P, x) &= \sum_{w \in \mathcal{L}(P)} x^{\text{maj}(w)} \sum_{m - \#D(w) \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0} x^{\lambda_1 + \cdots + \lambda_n} \\ &= \sum_{s=0}^m \sum_{m-s \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0} x^{\lambda_1 + \cdots + \lambda_n} \sum_{\substack{w \in \mathcal{L}(P) \\ \#D(w)=s}} x^{\text{maj}(w)} \\ &= \sum_{s=0}^n \begin{bmatrix} n+m-s \\ n \end{bmatrix}_x W_s(P, x), \end{aligned}$$

where the last equality follows from a result of Euler (see, e.g., [HW, Theorem 349]). Our result now follows upon replacing s by $m-s$ and noting that $W_s(P, x) = 0$ for $s > n$.

2. From the last equation we have

$$\begin{aligned} \sum_{m \geq 0} U_m(P, x) q^m &= \sum_{m \geq 0} \sum_{s \leq m} \begin{bmatrix} n+m-s \\ n \end{bmatrix}_x W_s(P, x) q^m \\ &= \sum_{s \geq 0} q^s W_s(P, x) \sum_{m-s \geq 0} \begin{bmatrix} n+(m-s) \\ n \end{bmatrix}_x q^{m-s} \\ &= \frac{\sum_{s \geq 0} q^s W_s(P, x)}{(1-q)(1-qx) \cdots (1-qx^n)}, \end{aligned}$$

where the last equality is also Euler's result (*ibid.*). ■

EXAMPLE. For $P = \{+e_2 - e_1, +e_2\}$ as before, we have

$$\mathcal{L}(P) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & +2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ +2 & +1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ +2 & -1 \end{pmatrix} \right\}$$

(with descents indicated by dots). Thus by Proposition 3.5, we have

$$\begin{aligned} F(P, x) &= \frac{x_{-1}}{(1-x_{-1})(1-x_2)} + \frac{x_2}{(1-x_2)(1-x_1x_2)} \\ &\quad + \frac{x_{-1}x_2}{(1-x_2)(1-x_{-1}x_2)} \end{aligned}$$

and

$$U(P, x) = \frac{x + x + x^2}{(1-x)(1-x^2)},$$

both of which agree (after a little manipulation) with our previous calculations. By the previous proposition,

$$U_m(P, x) = \left[\begin{matrix} n+m-1 \\ n \end{matrix} \right]_x (x^2 + 2x)$$

and

$$\sum_{m \geq 0} U_m(P, x) q^m = \frac{q(x^2 + 2x)}{(1-q)(1-qx)(1-qx^2)}.$$

The decomposition given by Theorem 3.3 also gives us a useful partitioning of the P -compatible chains.

PROPOSITION 3.7.

$$\{P\text{-compatible chains in } J(P) - \hat{0}\} = \coprod_{w \in \mathcal{L}(P)} \{\text{chains } c : c_R(w) \subseteq c \subseteq c_E(w)\}.$$

Proof. A chain c in $J(P) - \hat{0}$ is P -compatible

$$\begin{aligned} &\Leftrightarrow c = c(f) \quad \text{for some } f \in \mathcal{A}(P) \\ &\Leftrightarrow c = c(f) \quad \text{for some } f \in \mathcal{A}(w\Phi^+), w \in \mathcal{L}(P) \\ &\Leftrightarrow c_R(w) \subseteq c \subseteq c_E(w). \end{aligned}$$

The first equivalence is due to Proposition 3.1. The second equivalence is due to Theorem 3.3. The third equivalence is due to Proposition 3.4. ■

We can now give combinatorial interpretations to two invariants associated to P .

DEFINITION. Given a chain $c = I_1 < \dots < I_k$ in $\{+1, -1, 0\}$, we define the *rank set* S of c to be the set $S = \{\#I_1, \dots, \#I_k\}$ (where if $I_i = (\varepsilon_1, \dots, \varepsilon_n)$, then $\#I_i = \#\{k : \varepsilon_k \neq 0\}$). For $S \subseteq \{1, 2, \dots, n\}$, define

$$\begin{aligned} \alpha_S(P) &= \# \{P\text{-compatible chains with rank set } S \text{ in } J(P) - \hat{0}\} \\ \beta_S(P) &= \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_T(P). \end{aligned}$$

$\beta_S(P)$ is sometimes called the *rank-selected Möbius invariant* of P (see [St1, Sect. 3.12]).

PROPOSITION 3.8.

$$\begin{aligned}\alpha_S(P) &= \# \{w \in \mathcal{L}(P) : D(w) \subseteq S\} \\ \beta_S(P) &= \# \{w \in \mathcal{L}(P) : D(w) = S\}.\end{aligned}$$

Proof. By the previous proposition, we have

$$\begin{aligned}\alpha_S(P) &= \sum_{w \in \mathcal{L}(P)} \# \{\text{chains } c \text{ with rank set } S : c_r(w) \subseteq c \subseteq c_E(w)\} \\ &= \sum_{w \in \mathcal{L}(P)} \begin{cases} 1 & \text{if } D(w) \subseteq S \\ 0 & \text{else} \end{cases} \\ &= \# \{w \in \mathcal{L}(P) : D(w) \subseteq S\}.\end{aligned}$$

The second assertion follows by inclusion-exclusion. ■

Our next result gives reciprocity formulas that hold between the various generating functions for P and $-P = w_0 P$, where

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ -1 & -2 & \cdots & -n \end{pmatrix}$$

is the *longest element* of B_n (see [Bo, Chap. VI, Sect. 1, Corollaire 3] for information about longest elements).

THEOREM 3.9 (Reciprocity). 1. $F(-P, x) = (-1)^n F(P, x) \big|_{x_i \rightarrow 1/x_{-i}}.$

2. $\beta_J(-P) = \beta_{S-J}(P).$

3. $W_s(-P, x) = x^{\binom{n+1}{2}} W_{n-s}(P, x^{-1}).$

4. $W(-P, x) = x^{\binom{n+1}{2}} W(P, x^{-1}).$

5. $U(-P, x) = (-1)^n U(P, x^{-1}).$

6. $U_{-m}(-P, x) = (-1)^n U_{m-1}(P, x^{-1}).$

Proof. 1. From Proposition 3.5, we have that

$$\begin{aligned}F(P, x) &= \sum_{w \in \mathcal{L}(P)} \frac{\prod_{s_i \in D(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{i=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_i})} \\ &= \sum_{w \in \mathcal{L}(P)} \frac{x_{w_1}^{c_1(w)} \cdots x_{w_n}^{c_n(w)}}{\prod_{i=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_i})},\end{aligned}$$

where $c_i(w) = \#D(w) \cap \{s_i, s_{i+1}, \dots, s_n\}$. Note that $\mathcal{L}(-P) = w_0 \mathcal{L}(P)$, and $c_i(w_0 w) = n + 1 - i - c_i(w)$. Thus we have

$$\begin{aligned} F(-P, x) &= \sum_{u \in \mathcal{L}(-P)} \frac{x_{u_1}^{c_1(u)} \cdots x_{u_n}^{c_n(u)}}{\prod_{i=1}^n (1 - x_{u_1} x_{u_2} \cdots x_{u_i})} \\ &= \sum_{w \in \mathcal{L}(P)} \frac{x_{(w_0 w)_1}^{c_1(w_0 w)} \cdots x_{(w_0 w)_n}^{c_n(w_0 w)}}{\prod_{i=1}^n (1 - x_{(w_0 w)_1} x_{(w_0 w)_2} \cdots x_{(w_0 w)_i})} \\ &= \sum_{w \in \mathcal{L}(P)} \frac{x_{-w_1}^{n-c_1(w)} x_{-w_2}^{n-1-c_2(w)} \cdots x_{-w_n}^{1-c_n(w)}}{\prod_{i=1}^n (1 - x_{-w_1} x_{-w_2} \cdots x_{-w_i})}. \end{aligned}$$

Multiplying numerator and denominator above by $\prod_{i=1}^n (x_{-w_1} x_{-w_2} \cdots x_{-w_i})^{-1}$ gives

$$F(-P, x) = \sum_{w \in \mathcal{L}(P)} \frac{x_{-w_1}^{-c_1(w)} x_{-w_2}^{-c_2(w)} \cdots x_{-w_n}^{-c_n(w)}}{\prod_{i=1}^n (x_{-w_1}^{-1} x_{-w_2}^{-1} \cdots x_{-w_i}^{-1} - 1)} = (-1)^n F(P, x) \mid_{x_i \rightarrow 1/x_{-i}}.$$

2. This follows from the fact that $\beta_J(P) = \#\{w \in \mathcal{L}(P) : D(w) = J\}$, since it is easy to see that $\mathcal{L}(-P) = w_0 \mathcal{L}(P)$ and $D(w_0 w) = S - D(w)$.

3. Since $\#D(w_0 w) = \#(S - D(w)) = n - \#D(w)$, and $\text{maj}(w_0 w) = \binom{n+1}{2} - \text{maj}(w)$, we have

$$\begin{aligned} W_s(-P, x) &= \sum_{\substack{u \in \mathcal{L}(-P) \\ \#D(u) = s}} x^{\text{maj}(u)} \\ &= \sum_{w \in \mathcal{L}(P)} x^{\text{maj}(w_0 w)} \\ &= \sum_{\substack{w \in \mathcal{L}(P) \\ \#D(w) = n-s}} x^{\binom{n+1}{2} - \text{maj}(w)} \\ &= x^{\binom{n+1}{2}} W_{n-s}(P, x^{-1}). \end{aligned}$$

4. This follows from 3 and the fact that $W(P, x) = \sum_{s=0}^n W_s(P, x)$.

5. This follows from 1 and the fact that $U(P, x) = F(P, x) \mid_{x_{\pm i} \rightarrow x}$.

6. We have

$$\begin{aligned} U_{-m}(-P, x) &= \sum_{s=0}^n \begin{bmatrix} n-m-s \\ n \end{bmatrix}_x W_s(-P, x) \\ &= \sum_{s=0}^n \begin{bmatrix} n-m-s \\ n \end{bmatrix}_x x^{\binom{n+1}{2}} W_{n-s}(P, x^{-1}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{s=0}^n (-1)^n \begin{bmatrix} m+s-1 \\ n \end{bmatrix}_{x^{-1}} W_{n-s}(P, x^{-1}) \\
&= (-1)^n \sum_{s=0}^n \begin{bmatrix} m+n-s-1 \\ n \end{bmatrix}_{x^{-1}} W_s(P, x^{-1}) \\
&= (-1)^n U_{m-1}(P, x^{-1}),
\end{aligned}$$

where the third equality comes from the easy-to-check fact that

$$\begin{bmatrix} n-m-s \\ n \end{bmatrix}_x x^{\binom{n+1}{2}} = (-1)^n \begin{bmatrix} m+s-1 \\ n \end{bmatrix}_{x^{-1}},$$

and the fourth equality comes from replacing s by $n-s$. ■

We will explore one further counting function of P .

DEFINITION. The *order polynomial* $\Omega(P; m)$ of P is defined as

$$\Omega(P; m) = \# \mathcal{A}(P, m-1) = U_{m-1}(P; 1)$$

(its name anticipates the soon-to-be-proven fact that it is a polynomial in m). Define the *P-Eulerian numbers* $w_0(P), \dots, w_n(P)$ by

$$w_s(P) = \# \{w \in \mathcal{L}(P) : \# D(w) = s\} = W_s(P, 1).$$

For $1 \leq j \leq n$, define the numbers $e_j(P), e'_j(P)$ by

$$\begin{aligned}
e_j(P) &= \# \{f \in \mathcal{A}(P) : \{|f_i|\}_{i=1, \dots, n} = \{1, 2, \dots, j\}\} \\
e'_j(P) &= \# \{f \in \mathcal{A}(P) : \{|f_i|\}_{i=1, \dots, n} = \{0, 1, \dots, j-1\}\}.
\end{aligned}$$

PROPOSITION 3.10. 1. $\Omega(P; m) = \sum_{j=1}^n (e_j(P) \binom{m-1}{j-1} + e'_j(P) \binom{m-1}{j})$ and hence $\Omega(P; m)$ is a polynomial in m of degree n .

2. $\Omega(P; m) = \sum_{s=0}^n \binom{n+m-1-s}{n} w_s(P)$.
3. $\sum_{m \geq 0} q^m = (\sum_{s=0}^n w_s(P) q^{s+1}) / (1-q)^{n+1}$.
4. (Reciprocity) $\Omega(-P; m) = (-1)^n \Omega(P; -m+1)$.

Proof. 1. We consider two classes of $f \in \mathcal{A}(P; m-1)$: those f having $f_i \neq 0$ for all i , and those having some $f_i = 0$. For those f in the first class, if we know the set $R(f) = \{|f_i|\}_{i=1, \dots, n}$ has cardinality j , then there is a unique $f' \in \mathcal{A}(P)$ such that $R(f') = \{1, 2, \dots, j\}$ and $c(f) = c(f')$. Conversely, f' and $R(f)$ completely determine f , so there are $\sum_{j=1}^n e_j(P) \binom{m-1}{j-1}$ elements in the first class. Similarly, for f in the second class we must have $\{0\} \subseteq R(f) \subseteq \{0, 1, \dots, m-1\}$, and hence if $\# R(f) = j$ then there is a unique $f' \in \mathcal{A}(P)$ with $R(f') = \{0, 1, \dots, j-1\}$ and $c(f) = c(f')$. Again, f'

and $R(f)$ determine f , so there are $\sum_{j=1}^n e_j(P) \binom{m-1}{j-1}$ elements in the second class.

2. Plug $x = 1$ into Proposition 3.6, part 1.
3. Plug $x = 1$ into Proposition 3.6, part 2.
4. Plug $x = 1$ into Theorem 3.9, part 6. ■

EXAMPLE. Let $P = \{+e_2 - e_1, +e_2\}$, so $-P = \{-e_2 + e_1, -e_2\}$. One can check that

$$e_1(P) = 1, e_2(P) = 3, \quad e'_1(P) = e'_2(P) = 0$$

and

$$w_0(P) = w_2(P) = 0, \quad w_1(P) = 3.$$

Thus by the first part of the previous proposition, we have

$$\Omega(P; m) = 1 \binom{m-1}{1} + 3 \binom{m-1}{2} + 0 \binom{m-1}{0} + 2 \binom{m-1}{1} = 3 \binom{m}{2}$$

or by the second part of the previous proposition, we have

$$\begin{aligned} \Omega(P; m) &= \binom{2+m-1-0}{2} 0 + \binom{2+m-1-1}{2} 3 + \binom{2+m-1-2}{2} 0 \\ &= 3 \binom{m}{2}, \end{aligned}$$

so the two agree. To check a case of the third (reciprocity) part of the proposition, note that $w_0(-P) = w_2(-P) = 0$, $w_1(-P) = 3$, and hence $\Omega(-P; m) = 3 \binom{m}{2}$ also. Therefore we have

$$\begin{aligned} (-1)^2 \Omega(P, -m+1) &= 3 \binom{-m+1}{2} = \frac{3(-m+1)(-m)}{2} \\ &= \frac{3m(m-1)}{2} = \Omega(P; m), \end{aligned}$$

as expected.

4. THE LATTICES $\hat{J}(P)$

In this section, we take a closer look at the posets $J(P)$. Our goal is to show that they give a B_n -analogue of distributive lattices, by proving

an analogue of the *Fundamental Theorem of Distributive Lattices* [St1, Theorem 3.4.1], or *Birkhoff's theorem*.

DEFINITION. Let $\hat{J}(P)$ be the poset obtained from $J(P)$ by adjoining a (new) greatest element $\hat{1}$.

PROPOSITION 4.1. 1. $\hat{J}(P)$ is a sublattice of the lattice $\{+1, -1, 0\}^n$.

2. $\hat{J}(P)$ depends (up to lattice-isomorphism) only on the isomorphism class of P .

Proof. We prove 2 first. If $P \cong P'$, then $P = wP'$ for some $w \in B_n$ and hence we have

$$\langle f, \alpha \rangle \geq 0 \quad \forall \alpha \in P \Leftrightarrow \langle w(f), w(\alpha) \rangle \geq 0 \quad \forall \alpha \in P \Leftrightarrow \langle w(f), \beta \rangle \geq 0 \quad \forall \beta \in P'.$$

So w maps ideals of P onto ideals of P' , and since w is an automorphism of the order on $\{+1, -1, 0\}^n$, w is an order-isomorphism of $J(P)$ onto $J(P')$.

To prove 1, let \wedge, \vee denote meet and join in the lattice $\{+1, -1, 0\}^n$. We must show that if $g, f \in \hat{J}(P)$, then $g \wedge f, g \vee f \in \hat{J}(P)$. Clearly we may assume $g, f, g \wedge f, g \vee f$ are all unequal to $\hat{1}$. Given $\alpha \in P$, we want to show that $\langle \alpha, g \rangle \geq 0$ imply that $\langle \alpha, g \wedge f \rangle \geq 0$ and $\langle \alpha, g \vee f \rangle \geq 0$. We may assume α is of the form $+e_i$ or $+e_i - e_j$, since otherwise we could apply an element w of B_n to make it of this form, and use assertion 2.

If $\alpha = +e_i$, then $g_i, f_i \in \{+1, 0\}$, and hence $(g \wedge f)_i, (g \vee f)_i \in \{+1, 0\}$. Thus we have $\langle \alpha, g \wedge f \rangle \geq 0$ and $\langle \alpha, g \vee f \rangle \geq 0$, as desired.

If $\alpha = +e_i - e_j$, then the possibilities for $(g_i, g_j), (f_i, f_j)$ are exactly the vectors shown in Fig. 5. One can see that the vectors in Fig. 5 are closed under meets, and also closed under joins whenever their join is unequal to $\hat{1}$. Hence as long as $g \vee f \neq \hat{1}$ we have $\langle \alpha, g \wedge f \rangle \geq 0$ and $\langle \alpha, g \vee f \rangle \geq 0$, as desired. ■

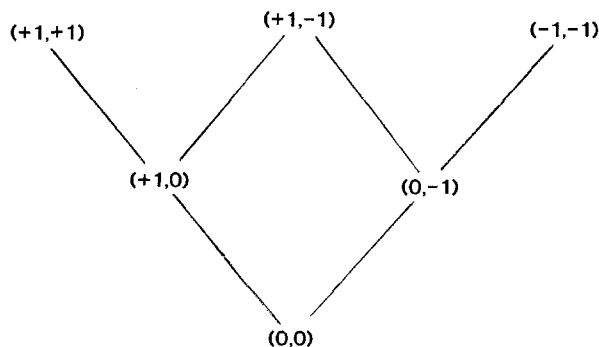


FIG. 5. Possibilities for (g_i, g_j) if $\langle +e_i - e_j, g \rangle \geq 0$.

We now assemble some properties of the lattices $\hat{J}(P)$ that will help us to characterize them intrinsically.

PROPOSITION 4.2. $J(P)$ is locally distributive, i.e., any interval $[x, y]$ in $J(P)$ is distributive.

Proof. It is easy to see that any interval in $\{+1, -1, 0\}^n$ is a Boolean algebra and hence distributive. Since an interval $[x, y]$ in $J(P)$ is a sublattice of an interval in $\{+1, -1, 0\}^n$ (Proposition 4.1), it must also be distributive. ■

DEFINITION. Given a finite lattice L , let $G(L)$ be the graph whose vertex set is the maximal elements of $L - \hat{1}$, and having an edge between two vertices m_1 and m_2 if m_1, m_2 both cover $m_1 \wedge m_2$ in L .

PROPOSITION 4.3. $G(\hat{J}(P))$ is connected.

Proof. First, we claim that $f \in \hat{J}(P) - \hat{1} = J(P)$ is maximal if and only if every coordinate $f_i \neq 0$. To see this, assume $f_i = 0$ for some i . We may assume $P \subseteq \Phi^+$ by applying some element w^{-1} with $w \in \mathcal{L}(P)$. Then $f \in J(P)$ implies $f \in \mathcal{A}(P)$, so $f \in \mathcal{A}(w\Phi^+)$ for some $w \in \mathcal{L}(P)$ by Proposition 3.3. Let $f' = f + w(+e_{|w^{-1}(i)|})$, and note that $w(+e_{|w^{-1}(i)|}) = \pm e_i$, so $f'_i \in \{+1, -1\}$ and hence $f < f'$ in the order on $\{+1, -1, 0\}$. Furthermore, if $\alpha \in \Phi^+$, then

$$\begin{aligned} \langle w(\alpha), f' \rangle &= \langle w(\alpha), f \rangle + \langle w(\alpha), w(+e_{|w^{-1}(i)|}) \rangle \\ &= \langle w(\alpha), f \rangle + \langle \alpha, +e_{|w^{-1}(i)|} \rangle \geq 0. \end{aligned}$$

So $f' \in J(P)$ (since $P \subseteq w\Phi^+$), contradicting the maximality of f .

Now suppose f, g are two distinct maximal elements of $J(P)$, and we will show that there is a path in $G(\hat{J}(P))$ connecting them. By restricting attention to the coordinates where they differ, we can assume $f_i \neq g_i$ for $i = 1, \dots, n$, and by applying an element $w \in B_n$, we can assume $f = (+1, +1, \dots, +1)$, $g = (-1, -1, \dots, -1)$. This implies that P can only contain roots of the form $+e_i - e_j$, and thus P corresponds to a poset on $\{1, 2, \dots, n\}$ (in which $i <_P j$ when $+e_i - e_j \in P$). Let i be minimal in this poset, and let g' be the vector with all -1 's except for a $+1$ in the i th coordinate. Then g and g' cover $g \wedge g'$, and we have that f and g' differ in one fewer coordinate than f and g did. So by induction we can find such a path. ■

DEFINITION. An element $f \in J(P)$ is said to be *join-irreducible* (written $f \in \text{Irr}(J(P))$) if $f \neq 0$, and $f = x \vee y$ implies either $f = x$ or $f = y$. For

$1 \leq i \leq n$, if $-e_i \notin P$, let I^{+i} denote the least element $f \in J(P)$ having $f_i = +1$ (i.e., $I^{+1} = \bigwedge \{f \in J(P) : f_i = +1\}$). Define I^{-i} similarly.

PROPOSITION 4.4. $f \in \text{Irr}(J(P)) \Leftrightarrow f = I^{+i}$ or I^{-i} for some i .

Proof. (\Leftarrow) Suppose $f = I^{+i}$ for some i (the $f = I^{-i}$ case is identical, or apply $w = (\begin{smallmatrix} i \\ -i \end{smallmatrix})$). Then if $f = x \vee y$, either $x_i = +1$ or $y_i = +1$, so either $x \geq I^{+i} = f$ or $y \geq I^{+i} = f$.

(\Rightarrow) Suppose $f \in \text{Irr}(J(P))$. Let

$$T = \{+i : f_i = +1\} \cup \{-i : f_i = -1\}.$$

Clearly, $f \geq I^t \forall t \in T$, and $f \leq \bigvee_{t \in T} I^t$. Using the fact that f is join-irreducible, and induction, we have $f = I^t$ for some $t \in T$. ■

DEFINITION. Given a finite lattice L , and I_1, I_2 two join-irreducibles in $L - \hat{1}$, we will say $I_1 \sim I_2$ if there exist two maximal elements m_1, m_2 in $L - \hat{1}$ that are adjacent in $G(L)$ and satisfy $I_i \leq m_i$, but $I_i \not\leq m_1 \wedge m_2$ for $i = 1, 2$.

PROPOSITION 4.5. Let $I_1, I_2 \in \text{Irr}(J(P))$. Then $I_1 \sim I_2$ if and only if for some $i \in \{\pm 1, \dots, \pm n\}$ we have $I_1 = I^i$ and $I_2 = I^{-i}$.

Proof. (\Rightarrow) Given $I_1 \sim I_2$ and m_1, m_2 as in the previous definition, by applying some element $w \in B_n$, we may assume $m_1 = (+1, +1, +1, \dots, +1)$, $m_2 = (-1, +1, +1, \dots, +1)$. Then the conditions that $I_i \leq m_i$ but $I_i \not\leq m_1 \wedge m_2$ imply $I_1 = I^{+1}, I_2 = I^{-1}$.

(\Leftarrow) Given that I^i, I^{-i} both exist in $J(P)$, we must exhibit m_1, m_2 as in the above definition. Let

$$M_1 = \{m \in J(P) : m \text{ maximal, and } m \geq I^i\}$$

$$M_2 = \{m \in J(P) : m \text{ maximal, and } m \geq I^{-i}\}.$$

Since we saw (in the proof of Proposition 4.3) that every maximal element m in $J(P)$ has all non-zero coordinates, these two sets M_1, M_2 disjointly cover all the maximal elements of $J(P)$. Since $G(\hat{J}(P))$ is connected, there must exist a pair of elements $m_1 \in M_1, m_2 \in M_2$ such that m_1, m_2 are adjacent in $G(\hat{J}(P))$. It is easy to see that these m_1, m_2 satisfy the conditions of the definition for $I^i \sim I^{-i}$. ■

PROPOSITION 4.6. Suppose $I_1, I_2, I_3, I_4 \in \text{Irr}(J(P))$ satisfy $I_1 \sim I_2$ and $I_3 \sim I_4$. Then

$$I_1 \leq I_3 \Leftrightarrow I_2 \geq I_4.$$

Proof. From the previous proposition, we have $I_1 = I^i$, $I_2 = I^{-i}$, $I_3 = I^j$, $I_4 = I^{-j}$ for some $i, j \in \{\pm 1, \dots, \pm n\}$. But

$$I^i \leq I^j \Leftrightarrow -\operatorname{sgn}(j) e_{|j|} + \operatorname{sgn}(i) e_{|i|} \in P \Leftrightarrow I^{-i} \geq I^{-j}$$

so the result follows. ■

PROPOSITION 4.7. *Let $\{I_i\}_{i=1, \dots, m} \in \operatorname{Irr}(J(P))$. Then $\bigvee_{i=1}^m I_i = \hat{1}$ if and only if there exist some $k \in \{1, 2, \dots, n\}$ and $r, s \in \{1, \dots, m\}$ such that $I^{+k} \leq I_r$ and $I^{-k} \leq I_s$.*

Proof. $\bigvee_{i=1}^m I_i = \hat{1}$ if and only if there exists some $k \in \{1, 2, \dots, n\}$ such that $I_1(k), \dots, I_m(k)$ have no upper bound in the partial order $+1 > 0, -1 > 0$. This is equivalent to saying that there exist $r, s \in \{1, \dots, m\}$ with $I_r(k) = +1$, $I_s(k) = -1$, which is the same as $I^{+k} \leq I_r$, $I^{-k} \leq I_s$. ■

It turns out that Propositions 4.2, 4.3, 4.6, 4.7 characterize the lattices $\hat{J}(P)$.

DEFINITION. We will say that a finite lattice L is B_n -distributive if it satisfies the following four conditions:

1. $L - \hat{1}$ is locally distributive and
2. $G(L)$ is connected.
3. If $I_1, I_2, I_3, I_4 \in \operatorname{Irr}(L - \hat{1})$ satisfy $I_1 \sim I_2$ and $I_3 \sim I_4$, then we have

$$I_1 \leq I_3 \Leftrightarrow I_2 \geq I_4.$$

4. If $\{I_i\}_{i=1, \dots, m} \subseteq \operatorname{Irr}(L - \hat{1})$ and $\bigvee_{i=1, \dots, m} I_i = \hat{1}$, then there exist $I_0, I'_0 \in \operatorname{Irr}(L - \hat{1})$ and $r, s \in \{1, \dots, m\}$ such that $I_0 \sim I'_0$ and $I_0 \leq I_r$, $I'_0 \leq I_s$.

We have not mentioned how the number n (in the name B_n -distributive) enters the picture. However, it is easy to see that conditions 1 and 2 together imply that L is ranked, and then we require that n be equal to the rank of L .

THEOREM 4.8 (Signed Birkhoff's Theorem). *A finite lattice L is isomorphic to $\hat{J}(P)$ for some signed poset P if and only if L is B_n -distributive. Furthermore, P is determined by L up to isomorphism as a signed poset.*

Proof. (\Rightarrow) This is the content of Propositions 4.2, 4.3, 4.6, 4.7.

(\Leftarrow) Assume L is B_n -distributive. We give a procedure to extract a signed poset P from L with the property that $L \cong \hat{J}(P)$.

Let m_1, m_2, \dots, m_M be an ordering of the maximal elements in $L - \hat{1}$ such that for all $k \geq 1$, there exists an $l < k$ for which m_{k+1} is adjacent to m_l in

$G(L)$ (such an ordering exists since $G(L)$ is connected by condition 2). We will construct P by a sequence P_1, P_2, \dots, P_M of approximations.

The first approximation P_1 is defined as follows. Let $+e_i, +e_i + e_j \in P_1$ for all $1 \leq i < j \leq n$. We will "think" of m_1 as being the ideal $(+1, +1, \dots, +1)$ and label the elements of $\text{Irr}(L - \hat{1})$ underneath m_1 arbitrarily as I^{+1}, \dots, I^{+n} . Then we add $+e_i - e_j$ to P_1 if and only if $I^{+i} \leq I^{+j}$. This completes the construction of P_1 .

Having gotten to stage k and constructed P_k , we proceed inductively as follows. Let m_l be adjacent to m_{k+1} in $G(L)$. Since m_l and m_{k+1} are incomparable, there exist at least one element $I \in \text{Irr}(L - \hat{1})$ satisfying $I \leq m_l$ but $I \not\leq m_{k+1}$ and at least one $I' \in \text{Irr}(L - \hat{1})$ satisfying $I' \leq m_{k+1}$ but $I' \not\leq m_l$. Any two such I, I' will have $I \sim I'$ by definition. But condition 3 implies that a given join-irreducible J can have $J \sim J'$ for at most one element J' : if $J \sim J'$ and $J \sim J''$, then we have $J \leq J \Rightarrow J' \geq J''$ and vice versa, so $J' = J''$. Thus there is a *unique* pair of join-irreducibles I, I' such that $I \sim I', I \leq m_l, I' \leq m_{k+1}$. Since $I \leq m_l$, by induction, I has already been labelled I^i for some $i \in \{\pm 1, \dots, \pm n\}$. We then label I' as I^{-i} , and "think" of m_{k+1} as the ideal that differs from m_l exactly in the i th coordinate and nowhere else. We produce P_{k+1} from P_k by removing $+e_i$, and then removing $+e_i + e_j$ if and only if $I^{-i} \not\leq I^j$ for some previously labelled join-irreducible I^j . This completes the construction of P_k . Proceeding through all of the elements m_1, m_2, \dots, m_M yields our final approximation $P_M = P$.

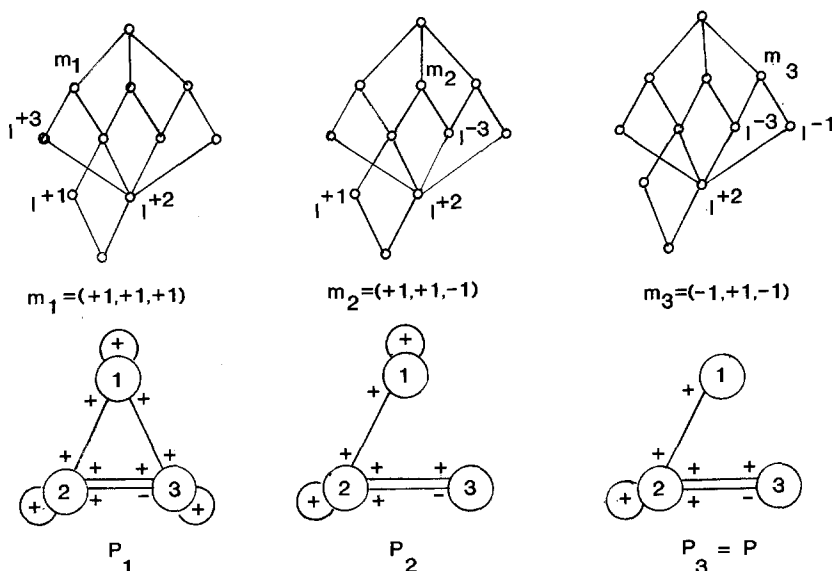


FIG. 6. Recovering a signed poset from a B_n -distributive lattice.

We want to show that $L \cong \hat{J}(P)$. The labelling of join-irreducibles during the above procedure gives a map $\phi: \text{Irr}(L - \hat{1}) \rightarrow \text{Irr}(J(P))$. A little thought shows that ϕ is a bijection (because of the way we removed roots of the form $+e_i$). Also, ϕ is a poset isomorphism, because of the way we included roots of the form $+e_i - e_j$ in P_1 (along with condition 3), and the way we removed roots of the form $+e_i + e_j$. We now use this bijection ϕ to define two maps $\tilde{\phi}: L \rightarrow \hat{J}(P)$ and $\tilde{\psi}: \hat{J}(P) \rightarrow L$. We let $\tilde{\phi}(\hat{1}) = \hat{1}$, and $\tilde{\phi}(x) = \bigvee_i \phi(I_i)$ if $x \neq \hat{1}$ and $x = \bigvee_i I_i$ is the unique irredundant decomposition of x into join-irreducibles (assured by the fact that the $[\hat{0}, x]$ is distributive). Similarly, we let $\tilde{\psi}(\hat{1}) = \hat{1}$, and $\tilde{\psi}(x) = \bigvee_i \phi^{-1}(I^i)$ if $f \neq \hat{1}$ and $f = \bigvee_i I^i$ is the unique irredundant decomposition of f . One can easily check that condition 4 implies that $\tilde{\phi}(x) = \hat{1}$ if and only if $x = \hat{1}$, and that $\tilde{\psi}(f) = \hat{1}$ if and only if $f = \hat{1}$. Also, since ϕ is a poset-isomorphism, $\tilde{\phi}$ and $\tilde{\psi}$ are inverse poset-isomorphisms. Hence $L \cong \hat{J}(P)$.

If $L \cong J(Q)$ for some other signed poset Q , we can produce an element $w \in B_n$ such that $wP = Q$ as follows. For $1 \leq i \leq n$, a given join-irreducible of L labelled I^{+i} during the above procedure must correspond to some join-irreducible I of $J(Q)$, and we know that I must in fact be of the form I^{w_i} for some $w_i \in \{\pm 1, \dots, \pm n\}$. Let $w = (\begin{smallmatrix} 1 & \dots & n \\ w_1 & \dots & w_n \end{smallmatrix})$, and it is not hard to see that $wP = Q$. ■

An example of the procedure in the preceding proof is shown in Fig. 6.

5. MORE ABOUT $\hat{J}(P)$

In this section, we investigate the interval structure of $\hat{J}(P)$ and compute its Möbius function and characteristic polynomial. We also give an EL-labelling (and hence a shelling) of a larger class of lattices which are hyperoctahedral analogues of *upper-semimodular* lattices.

PROPOSITION 5.1. *Let $[x, y]$ be an interval in $\hat{J}(P)$.*

1. *If $y = \hat{1}$, then $[x, y] \cong \hat{J}(P')$ for some signed poset P' .*
2. *If $y \neq \hat{1}$, then $[x, y] \cong J(P'')$ for some poset Q (where here $J(P'') = J(P''^+)$ is the usual lattice of order ideals in P'').*

Proof. 1. Given $[x, \hat{1}]$, let $T = \{i : x_i = 0\}$, and let P' be the induced signed subposet of P on T . Then the map from $[x, \hat{1}]$ to $\hat{J}(P')$ which ignores all coordinates outside T is clearly an isomorphism.

2. Given $[x, y]$, let $T = \{i : x_i \neq y_i\}$. Then by applying an element of B_n , we can make the restrictions of x and y to T look like $(0, \dots, 0)$ and $(+1, \dots, +1)$, respectively. Let P'' be the partial order on the numbers in T determined by $i <_{P''} j$ if $+e_i - e_j \in P$ and $i, j \in T$. Again, the map from

$[x, y]$ to $J(P'')$ which ignores all coordinates outside T is clearly an isomorphism. ■

In light of the previous proposition, rather than looking at intervals, we can concentrate our attention on the structure of the *whole* distributive lattice $J(P)$ for posets P and the *whole* B_n -distributive lattice $\hat{J}(P)$ for signed posets P .

EXAMPLE. Let $P = \{+e_2 - e_1, +e_2 + e_3, +e_3\}$. Then the interval $[(0, 0, +1), \hat{1}]$ in $\hat{J}(P)$, along with $\hat{J}(P')$ (where $P' = \{+e_2 - e_1\}$), is shown in Fig. 7. The interval $[(0, 0, 0), (+1, +1, +1)]$, along with $J(P'')$, where P'' is the poset determined by $\{+e_2 - e_1\}$, is also shown in Fig. 7.

DEFINITION. Let L be a lattice with a least element $\hat{0}$, and a greatest element $\hat{1}$. L is *complemented* if $\forall x \in L \exists y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$ (y is called a *complement* of x). A minimal element of $L - \hat{0}$ is called an *atom*. L is called *atomic* if $\hat{1} = \bigvee_{\text{atoms } x} x$.

It is well known (see, e.g., [St1, remarks after Proposition 3.4.4]), that for posets P on n elements, the following are equivalent:

1. $J(P)$ is complemented.
2. $J(P)$ is atomic.
3. $J(P)$ is the Boolean algebra $\{1, 0\}^n$.

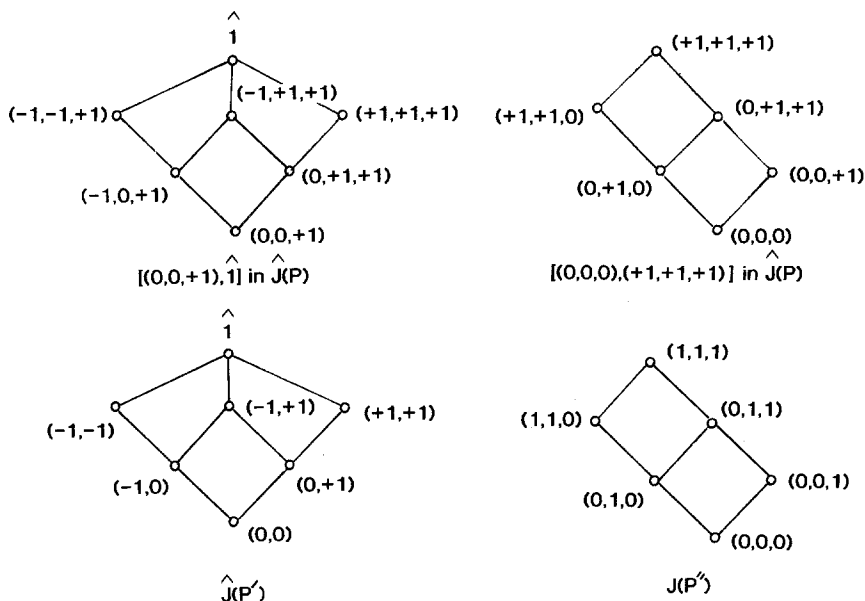


FIG. 7. Some examples of intervals in $\hat{J}(P)$.

PROPOSITION 5.2. *Let P be a signed poset. Then*

1. $\hat{J}(P)$ is complemented if and only if $\hat{J}(P)$ is the lattice $\{+1, -1, 0\}^n$, i.e., $P = \emptyset$.

2. $\hat{J}(P)$ is atomic if and only if some coordinate $i \in \{1, 2, \dots, n\}$ is vacuous in P , i.e., every $\alpha \in P$ has zero i^{th} coordinate.

Proof. 1. Clearly $\{+1, -1, 0\}^n$ is complemented, since the complement of f is given by $-f$. We must show then that $P \neq \emptyset$ implies $\hat{J}(P)$ is not complemented. Let $\alpha \in P$. Since $\hat{J}(P)$ only depends up to lattice-isomorphism on the isomorphism class of P , we can assume $P \subseteq \Phi^+$, so $\alpha \in \Phi^+$. If $\alpha = +e_i$, then one can check that I^{+i} has no complement in $\hat{J}(P)$. If $\alpha = +e_i + e_j$ or $+e_i - e_j$ then one can check that $I^{+i} \vee I^{+j}$ or $I^{+i} \vee I^{-j}$ has no complement in $\hat{J}(P)$, respectively.

2. By Proposition 4.7, $\hat{J}(P)$ is atomic if and only if for some $i \in \{1, 2, \dots, n\}$, both I^{+i} and I^{-i} are atoms. One can check that this means that i is vacuous in P . ■

DEFINITION. The Möbius function μ_Q of a poset Q is the map from the intervals of Q to \mathbf{Z} defined recursively as

$$\begin{aligned}\mu_Q(x, x) &= 1 & \forall x \in Q, \\ \mu_Q(x, y) &= - \sum_{z: x \leq z < y} \mu_Q(x, z).\end{aligned}$$

If Q is ranked with rank function r and has a least element $\hat{0}$, then the characteristic polynomial $\chi(Q, \lambda)$ is defined by

$$\chi(Q, \lambda) = \sum_{x \in Q} \mu_Q(\hat{0}, x) \lambda^{r(Q) - r(x)}.$$

See [Ro] for more on these definitions.

It is known [Cr, Corollary to Theorem 3] that for a finite lattice L , $\mu_L(\hat{0}, \hat{1}) = 0$ unless L is complemented. Hence for a poset P , we have

$$\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = \begin{cases} (-1)^n & \text{if } P \text{ is an antichain} \\ 0 & \text{else.} \end{cases}$$

PROPOSITION 5.3. *If P is a signed poset, then*

$$\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = \begin{cases} (-1)^n & \text{if } P = \emptyset \\ 0 & \text{else.} \end{cases}$$

Proof. If $P \neq \emptyset$, then P is not complemented by Proposition 5.2, so $\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = 0$. If $P = \emptyset$, then $\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = (-1)^{n-1}$, since $\hat{J}(P)$ is the poset

of faces of the n -dimensional *cross-polytope* (see [St1, Proposition 3.8.9]). Alternatively, one could compute this directly. ■

PROPOSITION 5.4. *Let P be a signed poset.*

1. *If $P = \emptyset$, then*

$$\chi(\hat{J}(P), \lambda) = \lambda(\lambda - 2)^n + (-1)^{n+1}.$$

2. *If $P \neq \emptyset$, let k be the number of coordinates i which are vacuous in P , and let a be the number of atoms of $\hat{J}(P)$. Then*

$$\chi(\hat{J}(P), \lambda) = \lambda^{n+1-a+k}(\lambda - 1)^{a-2k}(\lambda - 2)^k.$$

Proof. 1. Since $\hat{J}(P) = \{+1, -1, 0\}^n$, we can just compute directly. For any $x \in J(P)$ we have $\mu(\hat{0}, x) = (-1)^{r(x)}$, since $[\hat{0}, x]$ is a Boolean algebra of rank $r(x)$. There are $\binom{n}{i} 2^i$ elements of rank i in $\{+1, -1, 0\}$, and thus

$$\begin{aligned} \chi(\hat{J}(P), \lambda) &= \sum_{x \in J(P)} \mu(\hat{0}, x) \lambda^{n+1-r(x)} \\ &= (-1)^{n+1} + \sum_{i=0}^n (-1)^i \binom{n}{i} 2^i \lambda^{n+1-i} \\ &= \lambda(\lambda - 2)^n + (-1)^{n+1}. \end{aligned}$$

2. Our first observation is that

$$\chi(\hat{J}(P), \lambda) = \mu_{\hat{J}(P)}(\hat{0}, \hat{1}) + \lambda \cdot \chi(J(P), \lambda).$$

But $\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = 0$ by the previous proposition, so $\chi(\hat{J}(P), \lambda) = \lambda \cdot \chi(J(P), \lambda)$.

Next we note that $J(P)$ factors as a direct product of posets in the following manner. Let P' be the induced signed subposet of P on the set of $n - k$ non-vacuous coordinates. Then one easily sees that

$$J(P) = \{+1, -1, 0\}^k \times J(P').$$

Since it is easy to show that the characteristic polynomial satisfies

$$\chi(Q_1 \times Q_2, \lambda) = \chi(Q_1, \lambda) \chi(Q_2, \lambda),$$

we have

$$\begin{aligned} \chi(\hat{J}(P), \lambda) &= \lambda \chi(\{+1, -1, 0\}^k, \lambda) \chi(J(P'), \lambda) \\ &= \lambda \chi(\{+1, -1, 0\}, \lambda)^k \chi(J(P'), \lambda) \\ &= \lambda(\lambda - 2)^k \chi(J(P'), \lambda). \end{aligned}$$

It remains only for us to calculate $\chi(J(P'), \lambda)$. Since P' has no vacuous coordinates by construction, $\hat{J}(P')$ is not atomic (by Proposition 5.2). Hence if we let $z = \bigvee_{\text{atoms } x \in \hat{J}(P')} x$, then $z \neq \hat{1}$ so $z \in J(P)$. Since any interval $[\hat{0}, x]$ is distributive, $\mu(\hat{0}, x) = 0$ unless x is a join of atoms, i.e., unless $x \in [\hat{0}, z]$. Thus

$$\begin{aligned} \chi(J(P), \lambda) &= \sum_{x \in J(P')} \mu(\hat{0}, x) \lambda^{r(J(P')) - r(x)} \\ &= \sum_{x \in [\hat{0}, z]} \mu(\hat{0}, x) \lambda^{n-k-r(x)} \\ &= \lambda^{n-k-r(z)} \sum_{x \in [\hat{0}, z]} \mu(\hat{0}, x) \lambda^{r(z)-r(x)} \\ &= \lambda^{n-k-r(z)} \chi([\hat{0}, z], \lambda) \\ &= \lambda^{n-a+k} (\lambda - 1)^{a-2k}, \end{aligned}$$

where the last equality holds because $[\hat{0}, z]$ is a Boolean algebra of rank $a - 2k$. Thus, we have

$$\chi(\hat{J}(P), \lambda) = \lambda^{n+1-a+k} (\lambda - 1)^{a-2k} (\lambda - 2)^k. \quad \blacksquare$$

EXAMPLE. Let $n = 3$, and $P = \{+e_2 - e_3\}$. Then 1 is vacuous in P , so $k = 1$, and $a = \# \{(+1, 0, 0), (-1, 0, 0), (0, +1, 0), (0, 0, -1)\} = 4$. Thus by the previous proposition we have

$$\chi(\hat{J}(P), \lambda) = \lambda(\lambda - 1)^2 (\lambda - 2).$$

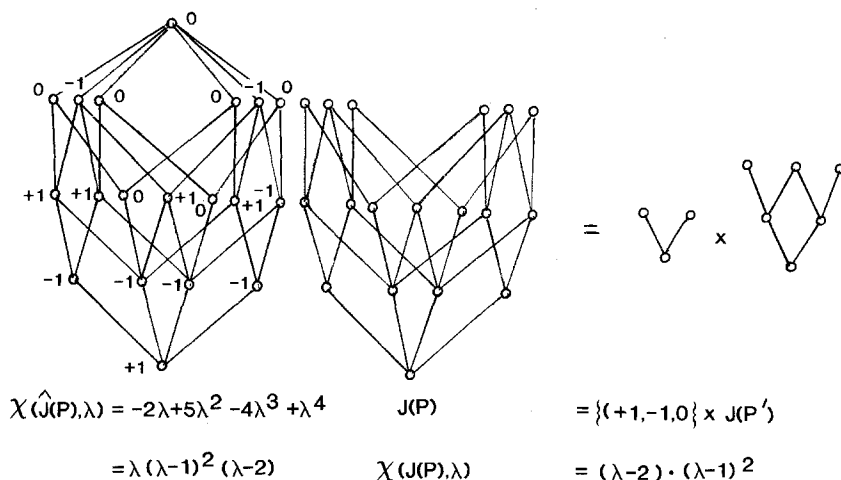
Figure 8 shows $\hat{J}(P)$ labelled with the values $\mu(\hat{0}, x)$, and the factorization $J(P) = \{+1, -1, 0\}^a \times J(P')$.

For a signed poset P , there is a condition on P which allows us to give the numbers $\beta_S(P)$ for $S \subseteq \{1, 2, \dots, n\}$ a Möbius function interpretation.

DEFINITION. We will say a signed poset P is *natural* if $P \subseteq \Phi^+$. For $S \subseteq \{1, 2, \dots, n\}$, let $\hat{J}(P)_S$ be the subposet of $\hat{J}(P)$ consisting of $\hat{0}$, $\hat{1}$, and all ideals I such that $\#I \in S$.

PROPOSITION 5.5. For P a natural signed poset and $S \subseteq \{1, 2, \dots, n\}$, we have

$$\beta_S(P) = (-1)^{\#S} \mu_{\hat{J}(P)_S}(\hat{0}, \hat{1}).$$

FIG. 8. An example of $\chi(J(P), \lambda)$.

Proof. We have that

$$\begin{aligned} \alpha_S(P) &= \# \{P\text{-compatible chains in } J(P) - \hat{0} \text{ with rank set } S\} \\ &= \# \{\text{chains in } J(P) - \hat{0} \text{ with rank set } S\} \end{aligned}$$

since P is natural. Hence

$$\begin{aligned} \beta_S(P) &= \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_L(P) \\ &= \sum_{T \subseteq S} (-1)^{\#(S-T)} \# \{\text{chains in } J(P) - \hat{0} \text{ with rank set } T\} \\ &= (-1)^{\#S} \sum_{\text{chains } c \in J(P)_S} (-1)^{\#c} \\ &= (-1)^{\#S} \mu_{J(P)_S}(\hat{0}, \hat{1}), \end{aligned}$$

where the last equality is by P. Hall's theorem ([Ro, Proposition 6], [St1, Proposition 3.8.5]). ■

COROLLARY 5.6. $(-1)^{\#S} \mu_{J(P)_S}(\hat{0}, \hat{1}) = \# \{w \in B_n : D(w) = S\}$ and hence is non-negative, for all $S \subseteq \{1, 2, \dots, n\}$.

Proof. Combine the previous proposition with Proposition 3.8. ■

The previous corollary is sometimes phrased as follows: the Möbius function of $\hat{J}(P)_S$ alternates in sign. We now show that there is an even

larger class of posets (containing all B_n -distributive lattices) with this property.

DEFINITION. We will say a finite lattice L is B_n -semimodular if L satisfies conditions 2, 3, and 4 in the definition of B_n -distributive, along with the following condition (which is weaker than the condition of local distributivity): every interval in $L - \hat{1}$ is (*upper*)-semimodular (a lattice is *upper-semimodular* if whenever x covers $x \wedge y$ we have that $x \vee y$ covers y).

We will make use of the notion of an EL-labelling [Bj].

DEFINITION. Let Q be a ranked poset. Write $x < \cdot y$ if y covers x in Q . We say Q is *edgewise-lexicographically labellable* or *EL-labellable* if we can label the edges $E = \{(x, y) : x < \cdot y\}$ in the Hasse diagram using a map $\lambda: E \rightarrow A$ to a linearly ordered set A satisfying:

1. For any interval $[x, y] \in Q$, there is a unique maximal chain

$$c[x, y] : x = x_0 < \cdot x_1 < \cdot \cdots < \cdot x_{k-1} < \cdot x_k = y,$$

for which the sequence of labels

$$(\lambda(x_0, x_1), \lambda(x_1, x_2), \dots, \lambda(x_{k-1}, x_k))$$

is (weakly) increasing in A .

2. $c[x, y]$ is the least among all maximal chains of $[x, y]$ when we order them by their label sequences, using the lexicographic extension of A to A^k .

In [Bj], Björner shows that when Q is EL-labellable and has $\hat{0}, \hat{1}$, for any subset S of the rank set of Q , the Möbius function of Q_S alternates in sign for the following reason:

$$(-1)^{\#S} \mu_{Q_S}(\hat{0}, \hat{1}) = \# \{ \text{maximal chains in } Q \text{ whose label set} \\ \text{decreases exactly after the ranks in } S \}.$$

It is known [Ga, Sect. 5] that all semimodular lattices are EL-labellable. We now prove a hyperoctahedral analogue of this result.

THEOREM 5.7. *If a finite lattice L is B_n -semimodular, then L is EL-labellable.*

Proof. We do the hyperoctahedral analogue of the proof of Theorem 5.1 in [Ga].

First we describe the edge-labelling λ . Let A be the following linear order on $\mathbf{Z} \cup \{\infty\} - \{0\}$:

$$+1 <_A +2 <_A +3 <_A \cdots <_A \infty <_A \cdots <_A -3 <_A -2 <_A -1.$$

Now pick a maximal element m in $L - \hat{1}$, and label the elements of $\text{Irr}(L - \hat{1})$ which lie under m by $I^{+1}, I^{+2}, \dots, I^{+k}$ in such a way that $I^{+i} \leq_L I^{+j}$ implies $i \leq j$. We extend this to all join-irreducibles as in the proof of Theorem 4.8, i.e., if $I \in \text{Irr}(L - \hat{1})$ and $I \sim I^{+i}$ for some i , then label I as I^{-i} (it is not hard to check that *all* join-irreducibles are labelled in this way). Now given $x < \cdot y$ in L , we label the edge (x, y) in the Hasse diagram with $\lambda(x, y)$ defined as

$$\lambda(x, y) = \begin{cases} \infty & \text{if } y = \hat{1} \\ \min_A \{i : x \vee I^i = y\} & \text{else.} \end{cases}$$

Before we show that this is an EL-labelling, we note one property of our labelling of $\text{Irr}(L - \hat{1})$: if $I^i \leq_L I^j$ then $i \leq_L j$. To see this, we check cases:

Case 1. i, j both positive. Then $I^i \leq_L I^j$ implies $i \leq_A j$ by construction.

Case 2. i, j both negative. Then $I^i \leq_L I^j$ implies $I^{-i} \geq_L I^{-j}$ by condition 3 of B_n -semimodularity, which implies $-i \geq -j$ and hence $i \leq_A j$.

Case 3. i positive, j negative. Then $i \leq_A j$ anyway.

Case 4. i negative, j positive. Then $I^i \leq I^j$ implies $I^i \leq m$, which contradicts the construction, so this case never happens.

Now we show that it is an EL-labelling. Let $x \leq y$ in L . We must exhibit $c[x, y]$, and show that it satisfies the two properties in the definition. If $x < \cdot y$, then $c[x, y]$ is just $x < \cdot y$, which trivially satisfies the definition. Otherwise, we will show how to construct $c[x, y]$ by induction on the length of the interval $[x, y]$.

Let

$$i = \min_A \{j : I^j \not\leq x, I^j \leq y, \text{ and } I^j \vee x \neq \hat{1}\}$$

(if this set is empty, then $y = \hat{1}$ and $x < \cdot y$). We claim that $I^i \wedge x < \cdot I^i$. To see this, assume not, i.e., let I^k satisfy

$$I^i \wedge x < I^k \vee (I^i \wedge x) < I^i.$$

Then $k \in \{j : I^j \not\leq x, I^j \leq y, \text{ and } I^j \vee x \neq \hat{1}\}$, and $I^k < I^i$ implies $k <_A i$, contradicting the minimality of i . Thus $I^i \wedge x < \cdot I^i$, and using the fact that the interval $[\hat{0}, I^i \vee x]$ is semimodular, we conclude that $x < \cdot x \vee I^i$. Thus if we start our chain $c[x, y]$ with $x < \cdot x \vee I^i$, we can then continue by

induction (replacing x by $x \vee I^i$), and $c[x, y]$ will certainly be the lexicographically smallest maximal chain from x to y .

We must check that this $c[x, y]$ has increasing labels. This is clearly true by construction if $y \neq \hat{1}$, since at each stage, the edge $x < x \vee I^i$ is labelled i . If $y = \hat{1}$, then the labels are certainly increasing, until the last step, which is labelled ∞ . Thus it would suffice to show that i is always positive. To see this, suppose not, i.e., $i < 0$. Then $I^i \vee x \neq \hat{1}$ implies that $I^{-i} \not\leq x$. We also can infer that $I^{-i} \vee x \neq \hat{1}$, else by condition 4 of B_n -semimodularity there would be some l for which $I^{-l} \leq I^{-i}$ and $I^{+l} \leq x$ and we would get the contradiction $I^i \leq I^{+l} \leq x$. Thus $-i$ is also in the set $\{j : I^j \not\leq x, I^j \leq y, \text{ and } I^j \vee x \neq \hat{1}\}$, and we have $-i <_{\mathcal{A}} i$, contradicting the minimality of i .

Thus we have exhibited the lexicographically smallest chain from x to y and have shown that it has increasing labels. Now suppose c is some other maximal chain from x to y with increasing labels. It remains only to show that $c = c[x, y]$, which we will do by induction on the length of c . Let c be $x = x_0 < x_1 < \dots < x_t = y$, and let j be the unique index satisfying $I^i \not\leq x_j$, but $I^i \leq x_{j+1}$.

Case 1. $x_{j+1} \neq \hat{1}$. Then $x_j \vee I^i = x_{j+1}$. Thus by minimality of i , this edge of c must be labelled i . In order for c to have increasing labels, this must be the first edge of c , i.e., $j = 0$, $x = x_j$, $x_{j+1} = x \vee I^i$. So c and $c[x, y]$ agree in their first step, and we can apply induction on the length of c .

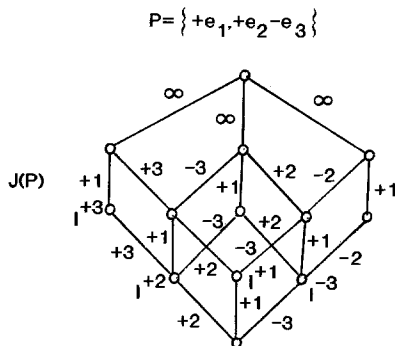
Case 2. $x_{j+1} = \hat{1}$. We will show that one of the labels on c between x and x_j is negative, and hence c is not increasing (since the last label on c is ∞).

To see this, let the labels on c between x and x_j be i_1, \dots, i_k . Let $x = I^{i_{k+1}} \vee \dots \vee I^{i_l}$ be an irredundant decomposition of x . Then we have

$$\begin{aligned} \hat{1} &= I^i \vee x_j = I^i \vee x \vee I^{i_1} \vee \dots \vee I^{i_k} \\ &= I^i \vee I^{i_1} \vee \dots \vee I^{i_l}. \end{aligned}$$

This implies (by condition 4 of B_n -semimodularity) that for some r, s we have $I^s \leq I^i$ and $I^{-s} \leq I^{i_r}$. Hence (by condition 3 of B_n -semimodularity) we have $I^{-i} \leq I^{-s} \leq I^{i_r}$, as long as I^{-i} exists in L . But I^{-i} must exist, since x_j is a maximal element of $L - \hat{1}$ which does not lie above I^i , so it must lie above I^{-i} (it is easy to see from condition 1 of B_n -semimodularity that every maximal element of $L - \hat{1}$ must lie above either I^i or I^{-i}). Now, if $r \geq k + 1$ then $I^{-i} \leq x$, contradicting the fact that $I^i \vee x \neq \hat{1}$. If $r \leq k$, then $I^{-i} \leq I^{i_r}$ implies that i_r is negative, as we desired. ■

Remark. One can check that for a natural signed poset P and $L = \hat{J}(P)$, if we choose m in the above proof to be the ideal $(+1, +1, \dots, +1) \in \hat{J}(P)$, and label each of the join-irreducibles I^{+i} as themselves (i.e., label the least

FIG. 9. An example of an EL-labelling for a lattice $J(P)$.

ideal having $+1$ in the i th coordinate as I^{+i}), then the label sequences of the maximal chains in $J(P)$ are exactly the same as $\mathcal{L}(P)$ (where we identify $w \in \mathcal{L}(P)$ with a sequence of numbers in $\{\pm 1, \dots, \pm n\}$).

EXAMPLE. Let $L = \hat{J}(P)$ for $P = \{+e_1, +e_2 - e_3\}$. Figure 9 shows an EL-labelling as in the proof above, and a listing of $\mathcal{L}(P)$.

A corollary of the previous theorem is the fact that finite B_n -semi-modular lattices are *shellable*, and hence *Cohen-Macaulay* (see [Bj] for the definitions and significance of these two conditions).

6. APPLICATIONS

Signed posets and their P -partitions may be used to derive the distributions of signed permutation statistics in the same way that posets and P -partitions are used to derive the distributions of permutation statistics (see [GG]). As an example, we give here a quick derivation of the generating function counting signed permutations $w \in B_n$ by the *major index* statistic $\text{maj}(w)$ (defined in Proposition 3.5). This may be viewed as an analogue of a special case of MacMahon's calculation of the generating function for permutations by their major index [Ma, Vol. I, No. 105; Vol. II, No. 453].

PROPOSITION 6.1. $\sum_{w \in B_n} x^{\text{maj}(w)} = (1+x)^n [n]!_x$, where

$$[n]!_x = \frac{1-x}{1-x} \frac{1-x^2}{1-x} \cdots \frac{1-x^n}{1-x}.$$

Proof. Let $P = \emptyset$, the empty signed poset. We will count $U(P, x)$ in two ways. On the one hand, by Proposition 3.5,

$$U(P, x) = \frac{\sum_{w \in B_n} x^{\text{maj}(w)}}{(1-x)(1-x^2) \cdots (1-x^n)}.$$

On the other hand,

$$\begin{aligned} U(P, x) &= \sum_{f \in \mathbb{Z}^n} x^{|f_1| + \cdots + |f_n|} \\ &= \left(\sum_{f \in \mathbb{Z}} x^{|f|} \right)^n \\ &= (\cdots x^3 + x^2 + x + 1 + x + x^2 + x^3 + \cdots)^n \\ &= \left(\frac{1+x}{1-x} \right)^n. \end{aligned}$$

Setting these two expressions for $U(P, x)$ equal to each other, we conclude that

$$\sum_{w \in B_n} x^{\text{maj}(w)} = (1-x)(1-x^2) \cdots (1-x^n) \left(\frac{1+x}{1-x} \right)^n = (1+x)^n [n]!_x. \quad \blacksquare$$

Another application of signed posets arises in a certain class of problems from invariant theory, which we will now describe. Let R be the ring $\mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ of finite Laurent polynomials in n variables with rational coefficients, and let B_n act on R in the following way: if $w = \begin{pmatrix} 1 & \cdots & n \\ w_1 & \cdots & w_n \end{pmatrix}$ then

$$w(x_i) = \begin{cases} x_{w_i} & \text{if } w_i > 0 \\ 1/x_{w_i} & \text{if } w_i < 0. \end{cases}$$

Given any subgroup $G \subseteq B_n$, one can ask for a description of the subring R^G of invariants under the action of G . Using the results in [Re, Sects. 4.1, 6.5], one can show that when G is a *reflection subgroup* of G , Theorem 5.7 may be used to write down an explicit basis for R^G as a free module over R^{B_n} . We refer the interested reader to [Re] for more details.

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